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On $p = 2$ generalized deformed parafermions and exotic statistics

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Abstract. Generalized deformed parafermions of order two are analysed according to the Lie (super)algebras they generate. A classification of the subtended exotic statistics is then given and, for each of them, remarkable embedded Lie structures are pointed out.

1. Introduction

Quantum groups and, more generally, nonlinear deformations of (the universal enveloping algebra of) Lie algebras have attracted a lot of attention in recent literature [1]. In particular, the generalized deformed oscillator algebra [2, 3] has been investigated in an intensive way due to the fact that it includes (besides others) the Bose, Fermi, parabose [4] and parafermi [4] quantization schemes as well as their q-analogues [5]. This algebra is characterized by the relations [2, 3]

$$[N, a] = -a \qquad [N, a^\dagger] = a^\dagger \tag{1.1}$$

$$a^\dagger a = F(N) \qquad aa^\dagger = F(N+1) \tag{1.2}$$

where $F(x)$ is any non-negative analytic function while N is a number operator. If we limit ourselves to the subcases

$$F(x) \geqslant 0 \text{ if } x \in \{1, 2, \ldots, p\} \qquad F(p+1) = 0 \tag{1.3}$$

where p is an arbitrary positive integer, we are dealing with the so-called generalized deformed parafermions [6]. The latter can easily be realized through the $(p+1)$-dimensional matrices [7]

$$a = \sum_{k=1}^{p} [F(p+1-k)]^{1/2} e_{k+1,k} \tag{1.4a}$$

$$a^\dagger = \sum_{k=1}^{p} [F(p+1-k)]^{1/2} e_{k,k+1} \tag{1.4b}$$

if

$$(e_{k,l})_{rs} = \delta_{rk}\delta_{sl}. \tag{1.5}$$

Here, we would like to concentrate on the first generalized context $p = 2$, the ordinary fermions (corresponding to $p = 1$) being, as has already been proved [8], non-deformable.

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Let us also mention that such a context is sufficiently rich for the introduction of different statistics (parastatistics [4], orthostatistics [9], genon statistics [10], ...).

In the non-deformed case (associated with $F(x) = 2$, $x = 1, 2$), the parafermionic annihilation and creation operators a and $a^\dagger$ are known [4] as the ladder generators of the compact Lie algebra $su(2, \mathbb{C})$. More precisely, they correspond to the finite-dimensional irreducible unitary representation $D^{(p/2)}$ (here with $p = 2$) of such an algebra. When the deformation is present, we obtain better results in the sense that we can put in evidence of either the previous $su(2, \mathbb{C})$ (when $F(1) = F(2)$ or when one of the $F(j)$'s vanishes) or the more extended $su(3, \mathbb{C})$ (in the other cases). In order to be convinced, one has just to calculate the commutation relations between the operators (1.4).

Being considered as odd operators [11], the ordinary $p = 2$-parafermionic operators satisfy remarkable *anti*commutation relations leading [12] to the Lie superalgebra $spl(2|1)$ isomorphic to the orthosymplectic structure $osp(2|2, \mathbb{R})$. Taking account of the deformation inside these operators and as can be verified in a straightforward manner by anticommuting the matrices (1.4), we can generalize this result to the Clifford algebra Cl_2, (if one of the $F(j)$'s vanishes) or, alternatively, to $osp(2|2, \mathbb{R})$.

Let us analyse these results in more detail.

2. Exotic statistics and the 'character reversal' phenomenon

Collating the information of the previous section, we are led to three different cases relevant to the Lie superalgebras generated by the generalized deformed parafermions given in (1.4) when $p = 2$:

$$
\begin{aligned}
&su(2, \mathbb{C}) \text{ and } Cl_2 \text{ if } F(1) = 0 \text{ or } F(2) = 0 \\
&su(2, \mathbb{C}) \text{ and } osp(2|2) \text{ if } F(1) = F(2) \\
&su(3, \mathbb{C}) \text{ and } osp(2|2) \text{ otherwise.}
\end{aligned}
$$

2.1. *The $F(1) = 0$ or $F(2) = 0$ case*

In this case, the generalized deformed parafermions satisfy the relations

$$a^2 = 0 \tag{2.1a}$$

$$a a^\dagger a = F(2) a \qquad \text{or} \qquad a a^\dagger a = F(1) a \tag{2.1b}$$

respectively, and coincide, in fact, with the orthofermions introduced by Mishra and Rajasekaran [9]. The so-called orthostatistics can, thus, be seen as a deformation of the parastatistics. If we want to look further into the physical meaning of these deformed operators, we have to consider the generalized deformed parasupercharges [7]

$$Q = \sum_{k=1}^{2} [\tfrac{1}{2} F(3-k)]^{1/2} (p_x + \mathrm{i} W_k(x)) e_{k+1,k} \tag{2.2a}$$

$$Q^\dagger = \sum_{k=1}^{2} [\tfrac{1}{2} F(3-k)]^{1/2} (p_x - \mathrm{i} W_k(x)) e_{k,k+1} \tag{2.2b}$$

where $p_x = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ and $W_k(x) (k = 1, 2)$ refer to the (para)superpotentials. Indeed, defining the subtended Hamiltonian H by the relation [7]

$$[Q, [Q^\dagger, Q]] = G(N) Q H \tag{2.3}$$

and its conjugated relation with

$$G(N) = 2F(N+1) - F(N) - F(N+2) \tag{2.4}$$

we obtain

$$H = \operatorname{diag}(H_1, H_2, H_3) \tag{2.5}$$

where

$$H_1 = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}W_1^2(x) + \tfrac{1}{2}W_1'(x) \tag{2.6a}$$

$$H_2 = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}W_1^2(x) - \tfrac{1}{2}W_1'(x) = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}W_2^2(x) + \tfrac{1}{2}W_2'(x) - \frac{c_1}{2} \tag{2.6b}$$

$$H_3 = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}W_2^2(x) - \tfrac{1}{2}W_2'(x) - \frac{c_1}{2} \tag{2.6c}$$

if $F(1) = 0$ (similar results being obtained if $F(2) = 0$) and where the prime stands for a derivative with respect to x. Limiting ourselves to the oscillator-like interaction $W_1(x) = \omega x$, we conclude through (2.6) that the Hamiltonian (2.5) describes a system consisting of a non-interacting three-level system [13] (being of Ξ-type if $W_2 = \omega x$ and of V-type if $W_2 = -\omega x$) and one bosonic mode.

Mathematically speaking, these orthostatistics and three-level systems are associated, as we have already mentioned, with the Lie structures $su(2,\mathbb{C})$ and Cl_2. We propose, in order to close this subsection, to point out the 'character reversal' phenomenon between these algebras. Such a 'character reversal' phenomenon is not new. It has appeared [14] between $osp(2m|2n)\square sh(2m|2n)$ and $osp(2m+1|2n+2)$ and consists of a correspondence between two Lie structures through the interchange of the *even* and *odd* characters of some non-trivial generators of these structures. The rigorous explanation of such a phenomenon is rather simple [14]: take one Lie algebra A and a Lie superalgebra S_1, they are related through the interchange of parity if you can find a Lie superalgebra S_2 containing both A and S_1. This S_2 is such that one of its odd generators (not a generator of S_1!)—let us call it γ—will commute with some of the even generators of A to yield some of the odd generators of S_1, *or* it will anticommute with the odd operators of S_1 to yield the corresponding even operators of A.

Let us illustrate this phenomenon on $su(2,\mathbb{C})$ and Cl_2. Both structures are included in $osp(3|2)$. Indeed, the positive even and odd roots of $osp(3|2)$ are [15]

$$\alpha_2 \qquad 2(\alpha_1+\alpha_2) \tag{2.7a}$$

and

$$\alpha_1 \qquad \alpha_1+\alpha_2 \qquad \alpha_1+2\alpha_2 \tag{2.7b}$$

respectively, where the subspace corresponding to the simple even root α_2 has basis $e_{1,3}-e_{3,2}$ and the subspace corresponding to the simple odd root α_1 has basis $e_{2,5}$. The information is complete if we recall that the Cartan subalgebra of $osp(3|2)$ is of dimension 2 (h_1 with basis $e_{4,4}-e_{5,5}$, h_2 with basis $e_{1,1}-e_{2,2}$). The subalgebra $su(2,\mathbb{C})$ evidently corresponds to h_2, e_{α_2} and $e_{-\alpha_2}$ while the Clifford algebra Cl_2 is isomorphic to the one generated by $e_{\alpha_1+2\alpha_2}$ and $e_{-\alpha_1-2\alpha_2}$. It is clear that the γ element belonging to $osp(3|2)$ and relating the even a and $a^\dagger$ (e_{α_a} and $e_{-\alpha_a}$) with the odd a and $a^\dagger$ ($e_{\alpha_1-2\alpha_2}$ and $e_{-\alpha_1-2\alpha_2}$) is

$$\gamma = e_{\alpha_1+\alpha_2} + e_{-\alpha_1-\alpha_2}. \tag{2.8}$$

2.2. The $F(1) = F(2)$ case

Considering this equality inside the operators (1.4), it is not difficult to be convinced that the following relations hold:

$$a^3 = 0 \tag{2.9a}$$

$$aa^\dagger a = F(1)a \tag{2.9b}$$

$$a^2a^\dagger + a^\dagger a^2 = F(1)a. \tag{2.9c}$$

We evidently recognize, up to a renormalization, the relations characterizing $p = 2$ parafermions and, so, the undeformed context. The corresponding Hamiltonian, coming from the constraint (2.3), and its conjugated relation coincides with the operator given in (2.5) and (2.6) but with $c_1 = 0$. With the oscillator-like interaction, this means that we are dealing with the (independent) superposition of a bosonic mode and a three-level system of V-type, only.

From a mathematical point of view, such a system is subtended by either the Lie algebra $su(2, \mathbb{C})$ or the Lie superalgebra $osp(2|2)$. In order to explain such an alternative, we again consider the superstructure $osp(3|2)$. The embedding of $osp(2|2)$ is evident through the four even elements

$$e_{2(\alpha_1+\alpha_2)} \qquad e_{-2(\alpha_1+\alpha_2)} \qquad h_1 \qquad h_2 \tag{2.10a}$$

and the four odd elements

$$e_{\alpha_1+2\alpha_2} \qquad e_{-\alpha_1-2\alpha_2} \qquad e_{\alpha_1} \qquad e_{-\alpha_1}. \tag{2.10b}$$

It is not difficult to convince ourselves that the γ element (2.8) is still convenient for relating $su(2, \mathbb{C})$ and $osp(2|2)$ through the 'character reversal' phenomenon, a and $a^\dagger$ being respectively associated with $\frac{1}{\sqrt{2}}(e_{\alpha_2} + e_{-\alpha_2})$, $\frac{1}{\sqrt{2}}(-e_{\alpha_2} + e_{-\alpha_2})$ (for $su(2, \mathbb{C})$) and $\frac{1}{\sqrt{2}}(e_{\alpha_1+2\alpha_2} - e_{\alpha_1})$, $\frac{1}{\sqrt{2}}(e_{-\alpha_1-2\alpha_2} + e_{-\alpha_1})$.

2.3. The other cases

A few specific trilinear relations arise between a and $a^\dagger$ but among them, some are interesting in connection with known statistics:

$$a^3 = 0 \tag{2.11a}$$

$$c_1a^2a^\dagger + c_2a^\dagger a^2 + (2c_2 - c_1)aa^\dagger a = F(1)(4c_2 - c_1)a \tag{2.11b}$$

if $F(2) = 2F(1)$ and c_1, c_2 are arbitrary constants. Indeed, if $F(1) = 1$ such relations are the ones characterizing the genons subtended by a recently introduced exotic statistics [10]. Another example is that for which $F(1) = \frac{4}{3}$ and $c_1 = c_2$: we then recognize in (2.11) the Rubakov–Spiridonov relations [16] implying that their parasupersymmetric quantum mechanics appears more exactly as a deformation of the parasupersymmetric quantum mechanics [17].

The Hamiltonian corresponding to the cases of this last subsection is associated with the description of a three-level system being of any type (Ξ, V or Λ).

Let us now point out the link between the Lie algebra $su(3, \mathbb{C})$ and the Lie superalgebra $osp(2|2)$, both structures being subtended by the exotic statistics relevant to this subsection. We consider the Lie (exceptional) superalgebra $G(3)$. The even part of this superalgebra is [15]

$$sp(2, \mathbb{R}) \oplus G(2) \tag{2.12}$$

the exceptional algebra $G(2)$ being generated by 14 even operators denoted by F_{pq} $(p, q = 1, 2, \ldots, 7)$ with

$$F_{pq} = -F_{qp} \tag{2.13}$$

$$\xi_{pqr} F_{pq} = 0 \tag{2.14a}$$

$$\xi_{pqr} = 1 \qquad \text{iff } \{p, q, r\} = (\{1,2,3\}, \{1,4,5\}, \{1,7,6\}, \{2,4,6\}, \{2,5,7\}, \{3,4,7\}, \{3,6,5\}). \tag{2.14b}$$

Their commutation relations are

$$[F_{pq}, F_{p'q'}] = 3\delta_{pp'} F_{qq'} - 3\delta_{qp'} F_{pq'} + 3\delta_{qq'} F_{pp'} - 3\delta_{pq'} F_{qp'} - \xi_{pqp''} \xi_{p'q'q''} F_{p''q''}. \tag{2.15}$$

This $G(2)$ is completed by $sp(2, \mathbb{R})(A_j, j = 1, 2, 3)$ and the odd part (14 generators) given by $Q_{kr} (k = 1, 2; r = 1, 2, \ldots, 7)$ with

$$[A_j, Q_{kr}] = -\mathrm{i}(a_j)_{k'k} Q_{k'r} \tag{2.16a}$$

$$[F_{pq}, Q_{kr}] = 2\delta_{pr} Q_{kq} - 2\delta_{qr} Q_{kp} - \eta_{pqrs} Q_{ks} \tag{2.16b}$$

$$\{Q_{kr}, Q_{k'r'}\} = -4\mathrm{i}\delta_{rr'} (c^2 a_j)_{kk'} A_j - \tfrac{1}{2} c^2_{kk'} F_{rr'} \tag{2.17}$$

where

$$a_j = \mathrm{i}\sigma_j \qquad c^2 = 2a_2 \qquad \sigma_j = \text{Pauli matrices} \tag{2.18a}$$

$$\eta_{pqrs} = \delta_{ps}\delta_{qr} - \delta_{pr}\delta_{qs} + \xi_{pqt}\xi_{rst}. \tag{2.18b}$$

The superalgebra $G(3)$ contains $su(3, \mathbb{C})$ through, for instance, the eight elements

$$F_{23} \qquad F_{45} \qquad F_{17} + 2F_{24} \qquad F_{16} - 2F_{25} \qquad F_{15} + 2F_{26} \qquad F_{14} - 2F_{27} \qquad F_{13} - 2F_{46} \qquad F_{12} + 2F_{47}. \tag{2.19}$$

In particular, the a and $a^\dagger$ leading to this $su(3, \mathbb{C})$ are, explicitly,

$$a = -\tfrac{1}{6}\sqrt{F(1)}(F_{17} + 2F_{24} - \mathrm{i}F_{16} + 2\mathrm{i}F_{25}) - \tfrac{1}{6}\sqrt{F(2)}(F_{13} - 2F_{46} + \mathrm{i}F_{12} + 2\mathrm{i}F_{47}) \tag{2.20a}$$

$$a^\dagger = \tfrac{1}{6}\sqrt{F(1)}(F_{17} + 2F_{24} + \mathrm{i}F_{16} - 2\mathrm{i}F_{25}) + \tfrac{1}{6}\sqrt{F(2)}(F_{13} - 2F_{46} - \mathrm{i}F_{12} - 2\mathrm{i}F_{47}). \tag{2.20b}$$

In what concerns the Lie superalgebra $osp(2|2)$, it is also included in $G(3)$ and is generated, in this case, by

$$A_1 \qquad A_2 \qquad A_3 \qquad F_{45} \tag{2.21a}$$

for the even part, and by

$$Q_{14} \qquad Q_{15} \qquad Q_{24} \qquad Q_{25} \tag{2.21b}$$

for the odd part. We can deduce from this $osp(2|2)$ the explicit forms of the associated (odd) a and $a^\dagger$:

$$a = -\tfrac{1}{2}\sqrt{F(1)}(Q_{24} + \mathrm{i}Q_{25}) + \tfrac{1}{2}\sqrt{F(2)}(Q_{24} - \mathrm{i}Q_{25}) \tag{2.22a}$$

$$a^\dagger = \tfrac{1}{2}\sqrt{F(1)}(Q_{14} - \mathrm{i}Q_{15}) + \tfrac{1}{2}\sqrt{F(2)}(Q_{14} + \mathrm{i}Q_{15}). \tag{2.22b}$$

The γ element relating expressions (2.20) and (2.22) defining the even and odd (respectively) generalized deformed parafermions a and $a^\dagger$ is

$$\gamma = \tfrac{1}{2}(Q_{12} - \mathrm{i}Q_{13} + Q_{16} - \mathrm{i}Q_{17} + Q_{22} + \mathrm{i}Q_{23} - Q_{26} - \mathrm{i}Q_{27}) \tag{2.23}$$

as can be verified. The two Lie structures $su(2, \mathbb{C})$ and $osp(2|2)$ are, thus, connected through the 'character reversal' phenomenon via the exceptional Lie superalgebra $G(3)$.

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